

Extended $f(R, L_m)$ gravity with generalized scalar field and kinetic term dependences

Tiberiu Harko^{1,*}, Francisco S.N. Lobo^{2,†} and Olivier Minazzoli^{3‡}

¹*Department of Physics and Center for Theoretical and Computational Physics,
The University of Hong Kong, Pok Fu Lam Road, Hong Kong*

²*Centro de Astronomia e Astrofísica da Universidade de Lisboa,
Campo Grande, Ed. C8 1749-016 Lisboa, Portugal and*

³*Jet Propulsion Laboratory, California Institute of Technology,
4800 Oak Grove Drive, Pasadena, CA 91109-0899, USA*

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We generalize previous work by considering a novel gravitational model with an action given by an arbitrary function of the Ricci scalar, the matter Lagrangian density, a scalar field and a kinetic term constructed from the gradients of the scalar field, respectively. The gravitational field equations in the metric formalism are obtained, as well as the equations of motion for test particles, which follow from the covariant divergence of the stress-energy tensor. Specific models with a nonminimal coupling between the scalar field and the matter Lagrangian are further explored. We emphasize that these models are extremely useful for describing an interaction between dark energy and dark matter, and for explaining the late-time cosmic acceleration.

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I. INTRODUCTION

Scalar fields play a fundamental role in the cosmological description of our Universe [1]. One of the first major extensions of general relativity, proposed by Dicke and Brans [2–4], conjectured that “Mach’s principle” might lead to a dependence of the local Newtonian gravitational constant, G , since in most cosmological models the total mass M and radius of curvature of the Universe $a(t)$ are related by an equation of the form $G^{-1} \sim M/a(t)c^2$. Consequently, in the variational principle of general relativity [2–4], it was proposed to substitute G^{-1} with a scalar field ϕ , and to also add to the action the kinetic energy corresponding to ϕ . Therefore, the variational principle of the Brans-Dicke theory can be formulated as $\delta \int (\phi R + L_m - \omega \nabla_\lambda \phi \nabla^\lambda \phi / \phi) \sqrt{-g} d^4x = 0$, where R is the scalar curvature, L_m is the matter Lagrangian, and ω is a coupling parameter. The scalar-tensor gravitational models have been intensively investigated, and can be considered as a valid approach in explaining the recent accelerated expansion of the Universe, inferred from the Type Ia supernova observations [5]. According to standard general relativity, the observed late-time cosmic acceleration can be successfully explained by introducing either a fundamental cosmological constant [6] – which would represent an intrinsic curvature of space-time – or a dark energy – which would *mimic* a cosmological constant (at least during the late stage of the cosmological evolution), as the concordance of observations are still in favor of the λ -CDM standard model [7], where λ is a constant. One of the currently main dark energy scenarios is based on the so-called quintessence [8], where dark

energy corresponds to a new scalar particle Φ .

After the Brans-Dicke proposal, other forms of scalar-tensor theoretical models were investigated. In particular, one can extend the coupling of the scalar field and curvature to a nonminimal coupling, provided by $F(\phi)R$, where, for example, $F(\phi) = 1 - \xi\phi^2$, and ξ is a coupling constant [9]. In these types of models non-minimally coupled Higgs field with a large coupling ξ might give rise to a successful inflation [9], which is otherwise very difficult to be achieved [10]. One of the positive features of Brans-Dicke-like theories is that they seem to be generically driven toward general relativity during the cosmological evolution of the Universe [11].

Couplings between the scalar and the matter fields have been investigated as well [12–20]. Indeed, such couplings generically appear in Kaluza-Klein theories with compactified dimensions [21] or in the low energy effective limit of string theories [12, 13, 15]. In the latter context, some authors proposed that the dilaton could be a good candidate for the quintessence [12] or the inflaton [13]. But even from a phenomenological point of view, it has been argued that specific restrictions such as gauge and diffeomorphism invariances essentially single out a particular set of effective theories which turns out to be Brans-Dicke-like theories with scalar/matter coupling [14]. A good feature of such theories is that under various assumptions – and similarly to Brans-Dicke-like theories without scalar/matter coupling – they seem to be driven toward a weak coupling through cosmological evolution [15]. Therefore, they seem to be able to explain the current tight constraints on the equivalence principle without fine-tuning parameters [16].

More recently, it has been argued that a scalar/matter coupling could be responsible for a dependency of the effective mass of the scalar-field upon the local matter density [22]. Such a scalar-field has been dubbed *chameleon* as “in regions of high density, the [scalar-field] chameleon

*Electronic address: harko@hkuc.hku.hk

†Electronic address: flobo@cii.fc.ul.pt

‡Electronic address: ominazzo@caltech.edu

blends with its environment and becomes essentially invisible to searches for EP violation and fifth force" [23].

The possibility of a nonminimal coupling between matter and geometry, somehow similar to the coupling between the scalar field and curvature in the Brans-Dicke theory, with the matter Lagrangian playing the role of ϕ , was considered in [24, 25], in the framework of the so-called $f(R)$ models of gravity [26–28], and further extended in [29]. An extension of the $f(R)$ gravity model, called $f(R, \phi)$ gravity was also proposed in [30], and further discussed and generalized in [31]. The action considered for the $f(R, \phi)$ gravity model is given by $S = \int [f(R, \phi)/2 - \omega(\phi)\nabla_\mu\phi\nabla^\mu\phi - V(\phi) + \beta L_m] \sqrt{-g} d^4x/\beta$, where β is a constant. In this context, the maximal extension of the standard Hilbert-Einstein action of general relativity, $S = \int (R/16\pi G + L_m) \sqrt{-g} d^4x$, was considered in [32], where the $f(R)$ type gravity models were maximally generalized by assuming that the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the matter Lagrangian L_m , so that $S = \int f(R, L_m) \sqrt{-g} d^4x$.

It is the purpose of the present paper to consider a generalized $f(R, L_m)$ type gravity model, in which, beyond the ordinary matter, described by its thermodynamic energy density ρ and pressure p , the Universe is filled with a scalar field ϕ . Consequently, in the gravitational action the scalar field, as well as its kinetic energy, $\nabla_\mu\phi\nabla^\mu\phi$ are also present, and the Lagrangian takes the form $L_{grav} = f(R, L_m, \phi, \nabla_\mu\phi\nabla^\mu\phi)$, where f is an arbitrary function. Therefore, in these models an explicit nonminimal coupling between matter and the scalar field is also allowed. As a first step in our study we obtain the gravitational field equations in the metric formalism, and the equations of motion for test particles, which follow from the covariant divergence of the stress-energy tensor. Some specific models with nonminimal scalar field-matter coupling are further explored.

II. GRAVITATIONAL FIELD EQUATIONS

We assume that the action for the modified theories of gravity, considered in this work, is given by

$$S = \int f(R, L_m, \phi, g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi) \sqrt{-g} d^4x, \quad (1)$$

where $\sqrt{-g}$ is the determinant of the metric tensor $g_{\mu\nu}$, and $f(R, L_m, \phi, g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi)$ is an arbitrary function of the Ricci scalar R , the matter Lagrangian density, L_m , a scalar field ϕ , and the gradients constructed from the scalar field, respectively. The only restriction on the function f is to be an analytical function of R , L_m , ϕ , and of the scalar field kinetic energy, respectively, that is, f must possess a Taylor series expansion about any point.

We define the “reduced” stress-energy tensor as [33]

$$\tau_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}. \quad (2)$$

By assuming that the matter Lagrangian density L_m depends only on the metric tensor components $g_{\mu\nu}$, and not on its derivatives, we obtain the stress-energy tensor as

$$\tau_{\mu\nu} = g_{\mu\nu}L_m - 2\frac{\delta L_m}{\delta g^{\mu\nu}}. \quad (3)$$

Furthermore, we assume that the scalar field ϕ is independent of the metric, i.e., $\delta\phi/\delta g^{\mu\nu} \equiv 0$. In the following we will denote, for simplicity, $(\nabla\phi)^2 = g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.

By varying the action S of the gravitational field with respect to the metric tensor components $g^{\mu\nu}$, we obtain the field equations of the $f[R, L_m, \phi, (\nabla\phi)^2]$ gravitational model as

$$f_R R_{\mu\nu} + (g_{\mu\nu}\nabla_\lambda\nabla^\lambda - \nabla_\mu\nabla_\nu) f_R - \frac{1}{2}(f - f_{L_m}L_m)g_{\mu\nu} = \frac{1}{2}f_{L_m}\tau_{\mu\nu} - f_{(\nabla\phi)^2}\nabla_\mu\phi\nabla_\nu\phi, \quad (4)$$

where the subscript of f denotes a partial derivative with respect to the arguments, i.e., $f_R = \partial f/\partial R$, $f_{L_m} = \partial f/\partial L_m$, $f_{(\nabla\phi)^2} = \partial f/\partial(\nabla\phi)^2$.

Now varying the action with respect to ϕ , provides the following evolution equation for the scalar field

$$\square_{(\nabla\phi)^2}\phi = \frac{1}{2}f_\phi, \quad (5)$$

where $f_\phi = \partial f/\partial\phi$ and

$$\square_{(\nabla\phi)^2} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu} \left[f_{(\nabla\phi)^2} \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right], \quad (6)$$

is the generalized D’Alembert operator of $f(R, L_m, \phi, \nabla_\mu\phi\nabla^\mu\phi)$ gravity.

The contraction of Eq. (4) provides the following relation between the Ricci scalar R , the matter Lagrangian density L_m , the derivatives of the scalar field, and the trace $\tau = \tau^\mu_\mu$ of the “reduced” stress-energy tensor,

$$f_R R + 3\nabla_\mu\nabla^\mu f_R - 2(f - f_{L_m}L_m) = \frac{1}{2}f_{L_m}\tau - f_{(\nabla\phi)^2}\nabla_\mu\phi\nabla^\mu\phi. \quad (7)$$

By taking the covariant divergence of Eq. (4), we obtain for the covariant divergence of the “reduced” stress-energy tensor the following expression

$$\begin{aligned} \frac{1}{2}\nabla^\sigma(f_{L_m}\tau_{\mu\sigma}) &= \frac{1}{2}(L_m\nabla_\mu f_{L_m} - f_\Phi\nabla_\mu\Phi) \\ &+ f_{(\nabla\phi)^2}\nabla_\mu\Phi\nabla_\sigma\nabla^\sigma\Phi + \nabla_\mu\Phi\nabla^\sigma\Phi\nabla_\sigma f_{(\nabla\phi)^2}. \end{aligned} \quad (8)$$

This relationship was deduced by taking into account the following mathematical identities

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R, \quad (9)$$

$$(\nabla_\nu\square - \square\nabla_\nu)f_R = -(\nabla^\mu f_R)R_{\mu\nu}, \quad (10)$$

and used the fact that we consider torsion-free space-times such that:

$$[\nabla_\sigma \nabla_\epsilon - \nabla_\epsilon \nabla_\sigma] \psi = 0, \quad (11)$$

where ψ is any scalar-field. Now, using Eq. (6) we get

$$\nabla^\sigma (f L_m \tau_{\mu\sigma}) = L_m \nabla_\mu f L_m. \quad (12)$$

For $\phi \equiv 0$, Eqs. (4) reduce to the field equations of the $f(R, L_m)$ model considered in [32]. For $\Phi \neq 0$, one recovers the good conservation equations for either general relativity and Brans-Dicke-like scalar-tensor theories (with and without scalar/matter coupling [19]). For instance, the total Lagrangian of the simplest matter-scalar field-gravitational field theory, with scalar field kinetic term and a self-interacting potential $V(\phi)$ corresponds to the choice

$$f = \frac{R}{2} + L_m + \frac{\lambda}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi), \quad (13)$$

where λ is a constant. The corresponding field equations can be immediately obtained from Eqs. (4) as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \tau_{\mu\nu} - \lambda \nabla_\mu \phi \nabla_\nu \phi + \left[\frac{\lambda}{2} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + V(\phi) \right] g_{\mu\nu}. \quad (14)$$

The scalar field satisfies the evolution equation

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left[\sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right] = \frac{1}{\lambda} \frac{dV(\phi)}{d\phi}, \quad (15)$$

while the stress-energy tensor satisfies the conservation equation

$$\nabla^\sigma \tau_{\mu\sigma} = 0. \quad (16)$$

III. MODELS WITH NONMINIMAL MATTER-SCALAR FIELD COUPLING

As an example of the application of the formalism developed in the previous section, we consider a simple phenomenological model, in which a scalar field is non-minimally coupled to pressure-less matter with *rest mass density* ρ . For the action of the system, consider

$$S = \int \left[\frac{R}{2} - F(\phi) \rho + \lambda g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] \sqrt{-g} d^4 x, \quad (17)$$

where $F(\phi)$ is an arbitrary function of the scalar field that couples non-minimally to ordinary matter. The field equations for this model are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = F(\phi) \rho u_\mu u_\nu + 2\lambda \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right], \quad (18)$$

where u^μ is the four-velocity of the matter fluid. The scalar field satisfies the evolution equation

$$\square \phi = -\frac{1}{2\lambda} \frac{dF(\phi)}{d\phi} \rho, \quad (19)$$

where \square is the usual d'Alembert operator defined in a curved space. The total stress-energy tensor of the scalar field-matter system is given by

$$T_{\mu\nu} = F(\phi) \rho u_\mu u_\nu + 2\lambda \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right]. \quad (20)$$

Through the Bianchi identities, the covariant divergence of $T^{\mu\nu}$ must be zero, that is, $\nabla_\nu T^{\mu\nu} = 0$. In the following, we assume that the mass density current is conserved, i.e., $\nabla_\nu (\rho u^\nu) = 0$. Using the latter condition, and the mathematical identity (11), we obtain first

$$F(\phi) \rho (u^\nu \nabla_\nu u^\mu) + \rho u^\mu u^\nu \frac{dF(\phi)}{d\phi} \nabla_\nu \phi + 2\lambda (\nabla^\mu \phi) \square \phi = 0. \quad (21)$$

By eliminating the term $\square \phi$ with the help of Eq. (19), we obtain

$$u^\nu \nabla_\nu u^\mu + \left[\frac{d}{d\phi} \ln F(\phi) \right] (u^\mu u^\nu \nabla_\nu \phi - \nabla^\mu \phi) = 0. \quad (22)$$

Using the identity $u^\nu \nabla_\nu u^\mu \equiv \frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta$, where $\Gamma_{\alpha\beta}^\mu$ are the Christoffel symbols corresponding to the metric, the equation of motion of the test particles non-minimally coupled to an arbitrary scalar field takes the form

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta + \left[\frac{d}{d\phi} \ln F(\phi) \right] (u^\mu u^\nu \nabla_\nu \phi - \nabla^\mu \phi) = 0. \quad (23)$$

A particular model can be obtained by assuming that $F(\phi)$ is given by a linear function,

$$F(\phi) = \frac{\Lambda + 1}{2} \left[1 + \frac{1}{2} (\Lambda - 1) \phi \right], \quad (24)$$

where Λ is a constant. Then the equation of motion becomes

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta + (u^\mu u^\nu - g^{\mu\nu}) \nabla_\nu \ln \left[1 + \frac{\Lambda - 1}{2} \phi \right] = 0. \quad (25)$$

In order to simplify the field equations we adopt for λ the value $\lambda = -(\Lambda^2 - 1)/8$. Then Eq. (19), determining the scalar field, takes the simple form $\square \phi = \rho$.

The gravitational field equations take the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\Lambda + 1}{2} T_{\mu\nu}, \quad (26)$$

with the total stress-energy tensor given by

$$T_{\mu\nu} = \left[1 + \frac{\Lambda - 1}{2} \phi \right] \rho u_\mu u_\nu - \frac{\Lambda - 1}{2} \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right]. \quad (27)$$

For $\Lambda = 1$ we reobtain the general relativistic model for dust. Other possible choices of the function $F(\phi)$, such as $F(\phi) = \exp(\phi)$, can be discussed in a similar way.

A more general model can be obtained by adopting for the matter Lagrangian the general expression [34–36]

$$L_m = - \left[\rho + \rho \int \frac{dp(\rho)}{\rho} - p(\rho) \right], \quad (28)$$

where ρ is the rest-mass energy density, p is the thermodynamic pressure, which, by assumption, satisfies a barotropic equation of state, $p = p(\rho)$. By assuming that the matter Lagrangian does not depend on the derivatives of the metric, and that the particle matter fluid current is conserved [$\nabla_\nu (\rho u^\nu) = 0$], the Lagrangian given by Eq. (28) is the unique matter Lagrangian that can be constructed from the thermodynamic parameters of the fluid [36].

The gravitational field equations and the equation describing the matter-scalar field coupling are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = F(\phi) \epsilon u_\mu u_\nu - pg_{\mu\nu} + \lambda Q_{\mu\nu}, \quad (29)$$

$$\square\phi = \frac{1}{2\lambda} \frac{dF(\phi)}{d\phi} \epsilon, \quad (30)$$

where $Q_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \nabla_\lambda \phi \nabla^\lambda \phi g_{\mu\nu}$, and where the total energy density is $\epsilon = \rho + \rho \int dp/\rho - p$ [34, 36]. With the use of the conservation equation $\nabla_\nu (\rho u^\nu) = 0$, one obtains the equation of motion of massive test particles as

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta + (u^\mu u^\nu - g^{\mu\nu}) \nabla_\nu \ln \left[1 + \int \frac{dp}{\rho} \right] = 0, \quad (31)$$

The equation of motion (31) can also be derived from the variational principle $\delta \int \sqrt{1 + \int dp/\rho} \sqrt{g_{\mu\nu} u^\mu u^\nu} ds = 0$. Models with scalar field-matter coupling were considered in the framework of the Brans-Dicke theory [18], with the action of the model given by $S = \int [\phi R/2 + (\omega/\phi) \nabla_\mu \phi \nabla^\mu \phi + F(\phi) L_m] \sqrt{-g} d^4x$. Such models can give rise to a late time accelerated expansion of the Universe for very high values of the Brans-Dicke parameter ω . Other models with interacting scalar field and matter have been considered in [17]. We emphasize the the gravitational theory considered in this work generalizes all of the above models.

IV. CONCLUSION

In the present paper we have presented a novel gravitational theory where the Lagrangian is given by an arbitrary function of the Ricci scalar, matter Lagrangian, a scalar field and its kinetic term, respectively. The field equations for this model were obtained for the general case, and the divergence of the stress-energy tensor has

been computed. Specific models with a nonminimal interaction between the scalar field and ordinary matter were explored. These models can be extremely useful in describing the interaction between dark energy, modeled as a scalar field, and dark matter, with or without pressure, respectively. Moreover, they can provide a realistic description of the late expansion of the Universe, where a possible interaction between ordinary matter and dark energy cannot be excluded *a priori*¹. Work along these lines is presently underway and the cosmological consequences of the present theory will also be investigated in detail in a future publication.

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¹ Let us note that such an interaction may be a good alternative to dark matter [20].

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